Second semester 2006
Backpaper examination
M.Math. Ist year
Algebra II — B.Sury

Q 1.
If p, q, r are distinct primes, find the degree of the splitting field of $(X^2 - pq)(X^2 - qr)(X^2 - pr)$ over $\mathbf{Q}$.

Q 2.
If L/K is a separable algebraic extension such that $[K(\alpha) : K] \leq n$ for every $\alpha \in L$, prove that $[L : K] \leq n$.

Q 3.
Let $N : \mathbf{F}_{q^n} \to \mathbf{F}_q$ be the norm map for a finite extension of finite fields. Compute the number of elements of norm 1 in $\mathbf{F}_{q^n}$.

Q 4.
Prove that $\mathbf{C}$ contains uncountably many proper subfields isomorphic to it.

Q 5.
Determine the Galois group of $X^p - 2$ over $\mathbf{Q}$ for an odd prime p.

Q 6.
If K is any field and $f \in K[X]$ is irreducible, prove that all roots of f in any splitting field of f over K have the same multiplicity.

Q 7.
If A is a commutative ring with unity, I is an ideal, and M is a finitely generated A-module such that $M = IM$, prove that there exists $a \in I$ such that $(1 + a)M = (0)$.

Q 8.
If $B \supset A$ is an integral extension, and $Q_1 \subset Q_2$ are prime ideals of B lying over the same prime ideal of A, show that $Q_1 = Q_2$.

Q 9.
Let K be an algebraically closed field. Using the Noether normalisation lemma, or otherwise, prove that every maximal ideal of $K[X_1, X_2, \cdots, X_n]$ is of the form $(X_1 - a_1, \cdots, X_n - a_n)$ for some $a_i \in K$.